Chapter 3
Modulation Index Estimation

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Of the many possible modes available for communicating information over a coherent communication channel, one that is quite common is to allocate a portion of the total transmitted power P_t to a discrete carrier for purposes of carrier synchronization. In the case of binary phase-shift keying (BPSK) modulation, the simplest way to accomplish this is to employ a phase modulation index β other than 90 deg. When this is done, the fraction of power allocated to the discrete carrier becomes $P_c = P_t \cos^2 \beta$ with the remaining fractional power $P_d = P_t \sin^2 \beta$ available for data detection. When using this signaling mode, one must assure oneself that the power spectrum of the data modulation is such that it does not interfere with the extraction of the discrete carrier by an appropriate carrier-tracking loop such as a phase-locked loop (PLL). In other words, the discrete carrier should be inserted at a point where the power spectrum of the data modulation is minimum, preferably equal to zero. In the case of digital data, this rules out direct modulation of the carrier with a non-return-to-zero (NRZ) data stream whose spectrum is maximum at zero frequency, which at radio frequency (RF) would correspond to the carrier frequency. Instead one can first modulate the data onto a subcarrier whose frequency is selected significantly higher than the data rate so that the sidebands of the data modulation are sufficiently reduced by the time they reach the carrier frequency. Alternatively, one can use a data format such as biphase-L (Manchester coding), whose power spectrum is identically equal to zero at zero frequency, and directly modulate the carrier.

On other occasions it might be preferable to use a coherent communication mode where carrier synchronization is established directly from the data-bearing signal, e.g., using a Costas loop. In this case, none of the transmitted power is allocated to a discrete carrier, and the system is said to operate in a suppressedcarrier mode, which in the case of BPSK corresponds to $\beta = 90$ deg. Although a Costas loop operates with a less efficient performance (e.g., larger mean-squared phase-tracking error) than a PLL, it offers the advantage of not requiring a discrete carrier to lock onto, and thus all of the transmitted power can be used for the purpose of data detection.

Given that either of the transmission modes discussed above is possible, in the case of autonomous receiver operation it is essential to have a means of estimating the modulation index or, equivalently, the ratio of transmitted carrier to data power. In this chapter, in Section 3.1 we first pursue the maximum-likelihood (ML) estimation approach to estimating modulation index along with appropriate approximation of the nonlinearities that result to allow for practical implementations at low and high signal-to-noise ratio (SNR) scenarios. In Section 3.2, we consider modulation index estimation for the case where carrier synchronization has not yet been established, i.e., the carrier phase is random. Here the ML estimation problem is too difficult to handle analytically and so we propose an ad hoc scheme instead. Finally, in Section 3.3, we describe how this scheme may be applied when the modulation type, symbol timing, and data rate are also unknown.

3.1 Coherent Estimation

3.1.1 BPSK

We begin by considering BPSK modulation where the received signal is given in complex baseband by Eqs. (1-3) and (1-6), or in passband by

$$
\begin{aligned}
r(t) &= \sqrt{2P_t}\sin\left(\omega_c t + \beta\sum_{n=-\infty}^{\infty} c_n p(t-nT)\right) + n(t) \\
&= \sqrt{2P_t\cos^2\beta}\sin\omega_c t + \sqrt{2P_t\sin^2\beta}\sum_{n=-\infty}^{\infty} c_n p(t-nT)\cos\omega_c t + n(t) \\
&= \sqrt{2P_c}\sin\omega_c t + \sqrt{2P_d}\sum_{n=-\infty}^{\infty} c_n p(t-nT)\cos\omega_c t + n(t) \qquad \text{(3-1)}
\end{aligned}
$$

where, in addition to the aforementioned parameter definitions, $\{c_n\}$ is a binary sequence, which for our purposes may be treated as independent, identically distributed (iid) data taking on values ± 1 with equal probability; $p(t)$ is the pulse shape, also taking on values ± 1; ω_c is the carrier frequency in rad/s;

$1/T$ is the data (symbol) rate; and $n(t)$ is a bandpass additive white Gaussian noise (AWGN) source with two-sided power spectral density $N_0/2$ W/Hz. Based on the above AWGN model, then for an observation of K data intervals, the conditional probability of the received signal given the data and the modulation index is given by

$$\begin{aligned} p\big(r(t)|\{c_n\},\beta\big) &= \frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{1}{N_0}\int_0^{KT}\Big[r(t)-\sqrt{2P_c}\sin\omega_c t\right. \\ &\qquad \left. -\sqrt{2P_d}\sum_{n=-\infty}^{\infty}c_n p(t-nT)\cos\omega_c t\Big]^2 dt\right) \\ &= C\exp\left(\frac{2\sqrt{2P_c}}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt\right) \\ &\quad \times\exp\left(\frac{2\sqrt{2P_d}}{N_0}\int_0^{KT} r(t)\sum_{n=-\infty}^{\infty}c_n p(t-nT)\cos\omega_c t\,dt\right) \end{aligned} \tag{3-2}$$

where C is a constant that has no bearing on the modulation index estimation to be performed. With some additional manipulation, Eq. (3-2) can be put in the form

$$\begin{aligned} p\big(r(t)|\{c_n\},\beta\big) &= C\exp\left(\frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt\right) \\ &\quad \times\prod_{k=1}^{K-1}\exp\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}c_k\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt\right) \end{aligned} \tag{3-3}$$

Averaging over the iid data sequence gives what is referred to as the conditional-likelihood function (CLF), namely,

$$\begin{aligned} p\big(r(t)|\beta\big) &= C\exp\left(\frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt\right) \\ &\quad \times\prod_{k=1}^{K-1}\cosh\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt\right) \end{aligned} \tag{3-4}$$

Next, taking the logarithm of Eq. (3-4), we obtain the log-likelihood function (LLF)

$$\Lambda \triangleq \ln p\big(r(t)|\beta\big) = \frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt$$
$$+\sum_{k=0}^{K-1}\ln\cosh\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt\right) \quad (3\text{-}5)$$

where we have ignored the additive constant $\ln C$.

Finally, differentiating the LLF with respect to β and equating the result to zero, we get

$$\frac{d}{d\theta}\ln p\big(r(t)|\beta\big) = -\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt$$
$$+\sum_{k=0}^{K-1}\tanh\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt\right)$$
$$\times\frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt = 0 \quad (3\text{-}6)$$

from which the ML estimate of β, namely, $\hat{\beta}$, is the solution to the transcendental equation

$$\int_0^{KT} r(t)\sin\omega_c t\,dt =$$
$$\left(\cot\hat{\beta}\right)\sum_{k=0}^{K-1}\tanh\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt\right)$$
$$\times\frac{2\sqrt{2P_t}}{N_0}\int_{kT}^{(k+1)T} r(t)p(t-kT)\cos\omega_c t\,dt \quad (3\text{-}7)$$

In order to arrive at an estimation algorithm that is practical to implement, one must now make suitable approximations to the nonlinearity in Eq. (3-7) corresponding to low and high data detection SNR conditions. For large arguments, the hyperbolic tangent nonlinearity can be approximated as

$$\tanh x \cong \operatorname{sgn} x \tag{3-8}$$

Applying this approximation to Eq. (3-7), we arrive at the simple result

$$\cot \hat{\beta} = \frac{\int_0^{KT} r(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} \left| \int_{kT}^{(k+1)T} r(t) p(t - kT) \cos \omega_c t \, dt \right|} \tag{3-9}$$

which for rectangular pulses simplifies further to

$$\cot \hat{\beta} = \frac{\int_0^{KT} r(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} \left| \int_{kT}^{(k+1)T} r(t) \cos \omega_c t \, dt \right|} \tag{3-10}$$

The result in Eq. (3-10) is intuitively satisfying since, in the absence of noise, it becomes

$$\cot \hat{\beta} = \frac{\int_0^{KT} \sqrt{2P_c} \sin^2 \omega_c t \, dt}{\sum_{k=0}^{K-1} \left| \int_{kT}^{(k+1)T} c_k \sqrt{2P_d} \cos^2 \omega_c t \, dt \right|} = \frac{\sqrt{2P_c}(KT/2)}{\sqrt{2P_d}(KT/2)} = \sqrt{\frac{P_c}{P_d}} \tag{3-11}$$

For small arguments, the hyperbolic tangent nonlinearity can be approximated as

$$\tanh x \cong x \tag{3-12}$$

Applying this approximation to Eq. (3-7), we arrive at the simple result

$$\cos \hat{\beta} = \frac{N_0 \int_0^{KT} r(t) \sin \omega_c t \, dt}{2\sqrt{2P_t} \sum_{k=0}^{K-1} \left(\int_{kT}^{(k+1)T} r(t) p(t - kT) \cos \omega_c t \, dt \right)^2} \tag{3-13}$$

which for rectangular pulses simplifies further to

$$\cos\hat{\beta} = \frac{N_0 \int_0^{KT} r(t)\sin\omega_c t\,dt}{2\sqrt{2P_t}\sum\limits_{k=0}^{K-1}\left(\int_{kT}^{(k+1)T} r(t)\cos\omega_c t\,dt\right)^2} \tag{3-14}$$

Unfortunately, there is no guarantee that the right-hand side of Eq. (3-13) will be less than or equal to unity and thus a solution to this equation may not always exist.

3.1.2 *M*-PSK

For M-phase shift keying (M-PSK) modulation ($M > 2$), the received signal can be represented in complex baseband using Eqs. (1-3) and (1-7), or in passband by

$$r(t) = \sqrt{2P_c}\sin\omega_c t + \sqrt{2P_d}\cos\left(\omega_c t + \sum_{n=-\infty}^{\infty}\theta_n p(t-nT)\right) + n(t) \tag{3-15}$$

where $\theta_n = [2q_n + (1+(-1)^{M/2})/2]\pi/M$ is the data modulation for the nth M-PSK symbol, with independent and uniformly distributed $q_n \in \{0, 1, \cdots, M-1\}$. The CLF analogous to Eq. (3-3) now becomes

$$\begin{aligned} p\big(r(t)|\{\theta_k\},\beta\big) &= C\exp\left(\frac{2\sqrt{2P_c}}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt\right) \\ &\quad\times\prod_{k=0}^{K-1}\exp\left(\frac{2\sqrt{2P_d}}{N_0}\int_{kT}^{(k+1)T} r(t)\cos\big(\omega_c t + \theta_k p(t-kT)\big)\,dt\right) \end{aligned} \tag{3-16}$$

Once again averaging over the data symbols, then because of the symmetry of the constellation around the circle, i.e., for each phase value there is one that is π radians away from it, we obtain

$$p\big(r(t)|,\beta\big) = C\exp\left(\frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt\right)$$
$$\times\prod_{k=0}^{K-1}\frac{2}{M}\sum_{q=0}^{(M/2)-1}\cosh\left[\frac{2\sqrt{2P_t}\sin\beta}{N_0}\right.$$
$$\left.\times\int_{kT}^{(k+1)T} r(t)\cos\left(\omega_c t+\frac{(2q+1)\pi}{M}p(t-kT)\right)\,dt\right] \tag{3-17}$$

where we have artificially introduced the modulation index β to have the same meaning as in the BPSK case. Once again taking the logarithm of Eq. (3-17), we obtain the LLF

$$\Lambda \triangleq \ln p\big(r(t)|\beta\big) = \frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r(t)\sin\omega_c t\,dt + \sum_{k=0}^{K-1}\ln\frac{2}{M}$$
$$\times\sum_{q=0}^{(M/2)-1}\cosh\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_{kT}^{(k+1)T} r(t)\cos\left(\omega_c t+\frac{(2q+1)\pi}{M}p(t-kT)\right)\,dt\right) \tag{3-18}$$

Finally, differentiating Eq. (3-18) with respect to β and equating the result to zero results in the transcendental equation

$$\int_0^{KT} r(t)\sin\omega_c t\,dt =$$
$$\left(\cot\hat{\beta}\right)\sum_{k=0}^{K-1}\frac{\sum_{q=0}^{(M/2)-1}x_k(q)\sinh\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}x_k(q)\right)}{\sum_{q=0}^{(M/2)-1}\cosh\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}x_k(q)\right)} \tag{3-19}$$

$$x_k(q) \triangleq \int_{kT}^{(k+1)T} r(t)\cos\left(\omega_c t+\frac{(2q+1)\pi}{M}p(t-kT)\right)\,dt$$

whose solution is the ML estimate of the modulation index. As for the BPSK case, to get an implementable estimator we must invoke suitable approximations to the nonlinearities in Eq. (3-19).

For large arguments, the hyperbolic sine and cosine nonlinearities can be approximated as

$$\begin{aligned} \sinh x &\cong \frac{1}{2}\exp\left(|x|\right)\operatorname{sgn} x \\ \cosh x &\cong \frac{1}{2}\exp\left(|x|\right) \end{aligned} \tag{3-20}$$

from which we obtain

$$\int_0^{KT} r(t)\sin\omega_c t\,dt = \left(\cot\hat{\beta}\right)\sum_{k=0}^{K-1}\frac{\displaystyle\sum_{q=0}^{(M/2)-1}|x_k(q)|\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|\right)}{\displaystyle\sum_{q=0}^{(M/2)-1}\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|\right)} \tag{3-21}$$

Noting further that for large SNR the summations in Eq. (3-21) are dominated by their largest term, we can make the further simplification

$$\sum_{q=0}^{(M/2)-1}|x_k(q)|\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|\right)\cong |x_k(q)|_{\max}\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|_{\max}\right) \tag{3-22}$$

$$\sum_{q=0}^{(M/2)-1}\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|\right)\cong\exp\left(\frac{2\sqrt{2P_t}\sin\hat{\beta}}{N_0}|x_k(q)|_{\max}\right) \tag{3-23}$$

where

$$|x_k\left(q\right)|_{\max}\triangleq\max_q|x_k\left(q\right)| \tag{3-24}$$

Finally, applying Eq. (3-22) to Eq. (3-21) gives the desired simplified solution for the ML estimate of modulation index for M-PSK, namely,

$$\cot \hat{\beta} = \frac{\int_0^{KT} r(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} |x_k(q)|_{\max}} \tag{3-25}$$

$$= \frac{\int_0^{KT} r(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} \max_q \left| \int_{kT}^{(k+1)T} r(t) \cos \left(\omega_c t + \frac{(2q+1)\pi}{M} p(t-kT) \right) \, dt \right|} \tag{3-26}$$

which for rectangular pulses simplifies further to

$$\cot \hat{\beta} = \frac{\int_0^{KT} r\,(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} |x_k\,(q)|_{\max}} = \frac{\int_0^{KT} r\,(t) \sin \omega_c t \, dt}{\sum_{k=0}^{K-1} \max_q \left| \int_{kT}^{(k+1)T} r\,(t) \cos \left(\omega_c t + \frac{(2q+1)\pi}{M} \right) \, dt \right|} \tag{3-27}$$

For low SNR, we can apply the small argument approximations

$$\begin{aligned} \sinh &\cong x \\ \cosh x &\cong 1 \end{aligned} \tag{3-28}$$

Note that these approximations are consistent with the approximation of the hyperbolic tangent nonlinearity given in Eq. (3-12). Thus, applying the approximations in Eq. (3-28) to Eq. (3-19) results in the ML estimate

$$\begin{aligned} &\cos \hat{\beta} \\ &= \frac{N_0 \int_0^{KT} r\,(t) \sin \omega_c t \, dt}{2\sqrt{2P_t} \sum_{k=0}^{K-1} \frac{2}{M} \sum_{q=0}^{(M/2)-1} x_k^2\,(q)} \\ &= \frac{M N_0 \int_0^{KT} r\,(t) \sin \omega_c t \, dt}{4\sqrt{2P_t} \sum_{k=0}^{K-1} \sum_{q=0}^{(M/2)-1} \left(\int_{kT}^{(k+1)T} r\,(t) \cos \left(\omega_c t + \frac{(2q+1)\pi}{M} p\,(t-kT) \right) \, dt \right)^2} \end{aligned} \tag{3-29}$$

which has the same difficulty as that in the discussion following Eq. (3-14).

3.2 Noncoherent Estimation

In the noncoherent case, the modulation index estimate must be formed in the absence of carrier synchronization. For simplicity, we again begin the investigation for BPSK modulation. The received signal is again modeled as in Eq. (3-1) with the addition of an unknown (assumed to be uniformly distributed) carrier phase to both the discrete and data-modulated carriers. Thus, analogous to Eq. (3-4), we now have the CLF

$$p\big(r(t)|\beta,\theta_c\big) = C\exp\left(\frac{2\sqrt{2P_t}\cos\beta}{N_0}\int_0^{KT} r\left(t\right)\sin\left(\omega_c t\ +\theta_c\right)dt\right)$$

$$\times\prod_{k=1}^{K-1}\cosh\left(\frac{2\sqrt{2P_t}\sin\beta}{N_0}\int_{kT}^{(k+1)T} r\left(t\right)p\left(t-kT\right)\cos\left(\omega_c t\ +\theta_c\right)\,dt\right) \quad \text{(3-30)}$$

The next step would be to average over the uniformly distributed carrier phase, which is an analytically intractable task. Even after approximating the nonlinearities as was done in the coherent case, performing this average is still analytically intractable. Thus, we abandon our search for the ML estimate and instead propose the following ad hoc approach.

Consider demodulating the received signal of Eq. (3-1), including now the unknown carrier phase θ_c, with the in-phase (I) and quadrature (Q) carriers (arbitrarily assumed to have zero phase relative to the unknown carrier phase of the received signal)

$$\begin{aligned} r_c\left(t\right) &= \sqrt{2}\cos\omega_c t \\ r_s\left(t\right) &= \sqrt{2}\sin\omega_c t \end{aligned} \quad \text{(3-31)}$$

Then, the outputs of these demodulations become

$$\begin{aligned} y_c\left(t\right) = r\left(t\right)r_c\left(t\right) &= \sqrt{P_c}\sin\theta_c + \sqrt{P_d}\sum_{n=-\infty}^{\infty} c_n p\left(t-nT\right)\cos\theta_c + n_c\left(t\right) \\ y_s\left(t\right) = r\left(t\right)r_s\left(t\right) &= \sqrt{P_c}\cos\theta_c - \sqrt{P_d}\sum_{n=-\infty}^{\infty} c_n p\left(t-nT\right)\sin\theta_c + n_s\left(t\right) \end{aligned} \quad \text{(3-32)}$$

where

$$
\begin{aligned}
n_c(t) &= n(t)\left(\sqrt{2}\cos\omega_c t\right) \\
n_s(t) &= n(t)\left(\sqrt{2}\sin\omega_c t\right)
\end{aligned}
\tag{3-33}
$$

Integrating $y_c(t)$ and $y_s(t)$ over K symbol durations and summing the squares of these integrations gives

$$
\begin{aligned}
\left(\int_0^{KT} y_c(t)\,dt\right)^2 + \left(\int_0^{KT} y_s(t)\,dt\right)^2 &= (KT)^2 P_t \cos^2\beta \\
&+ (KT)^2 P_t \sin^2\beta \left(\frac{1}{K}\sum_{k=0}^{K-1}\frac{1}{T}\int_{kT}^{(k+1)T} c_k p(t-kT)\,dt\right)^2 + N_1(t)
\end{aligned}
\tag{3-34}
$$

where $N_1(t)$ is composed of $S \times N$ and $N \times N$ terms. For sufficiently large K, the data-dependent term becomes vanishingly small, in which case Eq. (3-34) simplifies to

$$
\left(\int_0^{KT} y_c(t)\,dt\right)^2 + \left(\int_0^{KT} y_s(t)\,dt\right)^2 = (KT)^2 P_t \cos^2\beta + N_1(t) \tag{3-35}
$$

Next, noting that the first terms in Eq. (3-32) are constant with time, form the difference signals

$$
\begin{aligned}
y_c(t) - y_c(t-T) &= \sqrt{P_t}\sin\beta \sum_{n=-\infty}^{\infty} (c_n - c_{n-1})\, p(t-nT)\cos\theta_c \\
&\quad + n_c(t) - n_c(t-T) \\
y_s(t) - y_s(t-T) &= -\sqrt{P_t}\sin\beta \sum_{n=-\infty}^{\infty} (c_n - c_{n-1})\, p(t-nT)\sin\theta_c \\
&\quad + n_s(t) - n_s(t-T)
\end{aligned}
\tag{3-36}
$$

Now first squaring these signals and then integrating them over K symbol durations, the sum of these integrations becomes

$$\int_0^{KT}\big(y_c(t)-y_c(t-T)\big)^2dt+\int_0^{KT}\big(y_s(t)-y_s(t-T)\big)^2dt=$$
$$P_t\sin^2\beta\sum_{k=0}^{K-1}\int_{kT}^{(k+1)T}\left(2-2c_kc_{k-1}\right)p^2\left(t-kT\right)dt+N_2\left(t\right)\quad(3\text{-}37)$$

where again $N_2(t)$ is composed of $S\times N$ and $N\times N$ terms. Once again, for sufficiently large K, the data-dependent term becomes vanishingly small and, assuming for convenience rectangular pulses, Eq. (3-37) simplifies to

$$\int_0^{KT}\big(y_c(t)-y_c(t-T)\big)^2dt+\int_0^{KT}\big(y_s(t)-y_s(t-T)\big)^2dt=$$
$$2KTP_t\sin^2\beta+N_2(t)\quad(3\text{-}38)$$

Finally then, from observation of Eqs. (3-35) and (3-38), it is reasonable to propose the ad hoc noncoherent estimator of modulation index

$$\cot\hat{\beta}=\sqrt{\frac{2\left[\left(\int_0^{KT}y_c(t)dt\right)^2+\left(\int_0^{KT}y_s(t)dt\right)^2\right]}{KT\left[\int_0^{KT}\big(y_c(t)-y_c(t-T)\big)^2dt+\int_0^{KT}\big(y_s(t)-y_s(t-T)\big)^2dt\right]}}\quad(3\text{-}39)$$

Clearly, in the absence of noise this estimator produces the true value of the modulation index. Also, it has an advantage over Eqs. (3-14) and (3-29) in that the SNR need not be known to compute it. The architecture given by Eq. (3-39) is shown in Fig. 3-1.

3.3 Estimation in the Absence of Knowledge of the Modulation, Data Rate, Symbol Timing, and SNR

The modulation index estimators in Section 3.1 do not require an SNR estimate, and the ones in Section 3.2 require neither an SNR estimate nor a carrier phase estimate. However, they both require explicit knowledge of the phase-shift keying (PSK) modulation size, data rate, and symbol timing, as seen by the use (either explicitly or implicitly) of the parameters M and T and precise integration limits in Eqs. (3-10), (3-14), (3-27), (3-29), and (3-39).

In this section, we extend the ad hoc modulation index estimator of Eq. (3-39) for BPSK signals to a general M-PSK modulation where M is un-

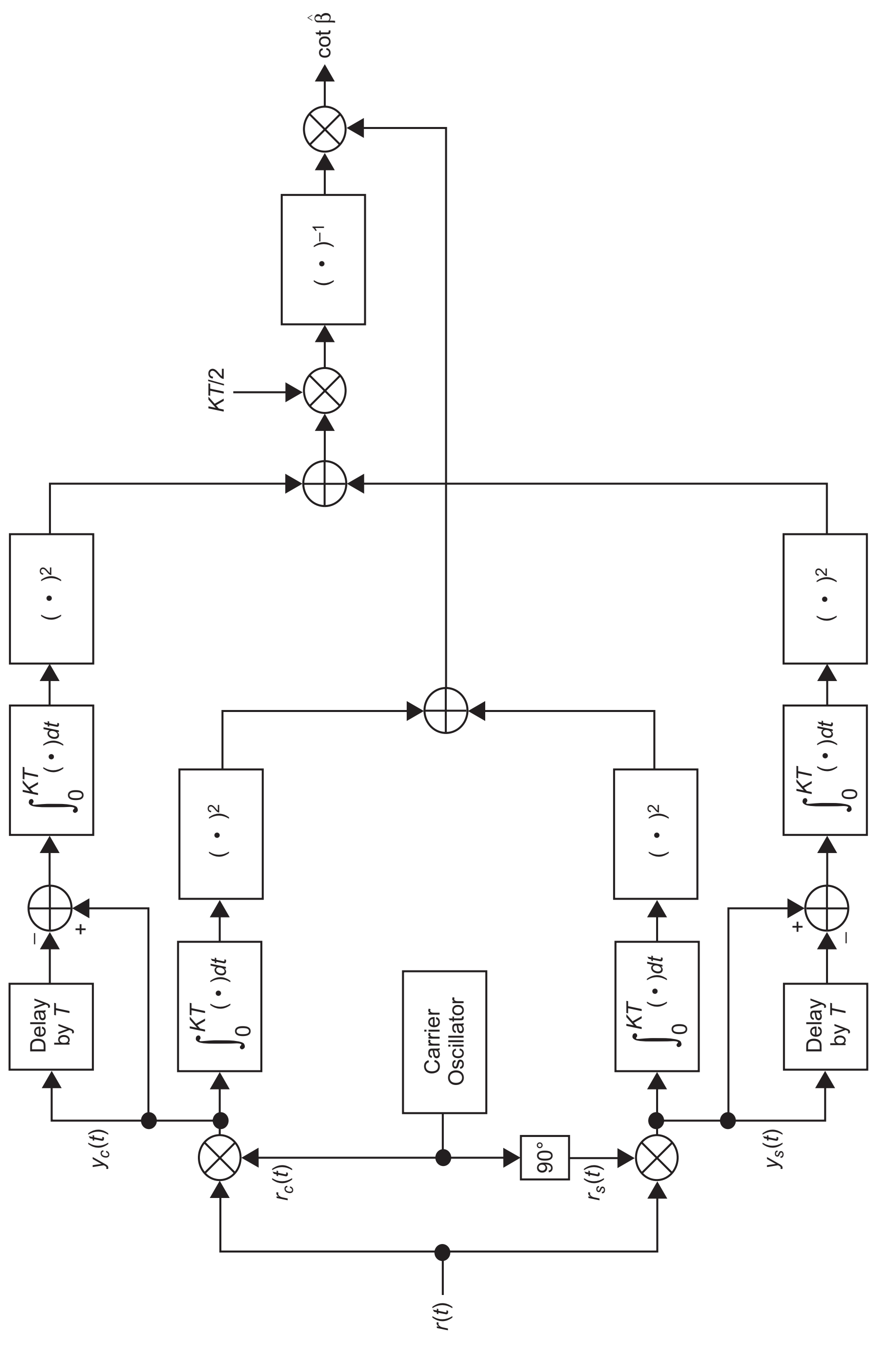

Fig. 3-1. An open-loop modulation index estimator.

known, and where the symbol rate ($1/T$) and fractional symbol timing (ε) are also unknown. We assume that T takes on values in a finite set $\mathcal{T}$, and we define

$$T^* \triangleq \max_{T}\{T \in \mathcal{T}\} \tag{3-40}$$

The received signal can be represented as

$$\begin{aligned} r(t) &= \sqrt{2P_c}\sin(\omega_c t + \theta_c) \\ &\quad + \sqrt{2P_d}\cos\left(\omega_c t + \theta_c + \sum_{n=-\infty}^{\infty}\theta_n p(t - nT - \varepsilon T)\right) + n(t) \end{aligned} \tag{3-41}$$

which is the same as Eq. (3-15) except that we have introduced unknown parameters θ_c and ε, and we now allow for the possibility of BPSK as well, so that $\theta_n = [2q_n + (1 + (-1)^{M/2})/2]\pi/M$ is the data modulation for the nth M-PSK symbol, with independent and uniformly distributed $q_n \in \{0, 1, \cdots, M-1\}$.

After mixing with in-phase and quadrature signals [see Eq. (3-31)], we have

$$\begin{aligned} y_c(t) = r(t)r_c(t) &= \sqrt{P_c}\sin\theta_c + \sqrt{P_d}\cos\left[\theta_c + \sum_{n=-\infty}^{\infty}\theta_n p(t - nT - \varepsilon T)\right] \\ &\quad + n_c(t) \\ y_s(t) = r(t)r_s(t) &= \sqrt{P_c}\cos\theta_c - \sqrt{P_d}\sin\left[\theta_c + \sum_{n=-\infty}^{\infty}\theta_n p(t - nT - \varepsilon T)\right] \\ &\quad + n_s(t) \end{aligned} \tag{3-42}$$

where $n_c(t)$ and $n_s(t)$ are described by Eq. (3-33), as before.

Following the same strategy for ad hoc estimation as in Section 3.2, we integrate over a long duration. In this case, the integration limits are not necessarily aligned to the symbols, and T^* is used in place of the (unknown) T, to obtain

$$\begin{aligned} \frac{1}{KT^*}\int_0^{KT^*} y_c(t)dt &= \sqrt{P_c}\sin\theta_c \\ &\quad + \frac{1}{KT^*}\int_0^{KT^*}\left[\sqrt{P_d}\cos\left(\theta_c + \sum_{n=-\infty}^{\infty}\theta_n p(t - nT - \varepsilon T)\right) + n_c(t)\right]dt \end{aligned} \tag{3-43}$$

and

$$\frac{1}{KT^*}\int_0^{KT^*} y_s(t)dt = \sqrt{P_c}\cos\theta_c$$
$$+\frac{1}{KT^*}\int_0^{KT^*}\left[\sqrt{P_d}\sin\left(\theta_c+\sum_{n=-\infty}^{\infty}\theta_n p(t-nT-\varepsilon T)\right)+n_s(t)\right]dt \qquad (3\text{-}44)$$

Using the facts that (1) M-PSK, for all M even, has the property that if θ_n is an allowable modulation angle, $\theta_n+\pi$ is as well, (2) each point in the signal constellation is equally likely, (3) both NRZ and Manchester satisfy $p(t)\in\{-1,1\}$ for all t, and (4) $\cos(\alpha)=-\cos(\alpha+\pi)$ and $\sin(\alpha)=-\sin(\alpha+\pi)$, it follows that the integrals in Eqs. (3-43) and (3-44) each approach zero as $K\to\infty$. Thus, for sufficiently large K, we may write

$$\left(\frac{1}{KT^*}\int_0^{KT^*} y_c(t)dt\right)^2 \cong P_c\sin^2\theta_c \qquad (3\text{-}45)$$

$$\left(\frac{1}{KT^*}\int_0^{KT^*} y_s(t)dt\right)^2 \cong P_c\cos^2\theta_c \qquad (3\text{-}46)$$

or

$$\left(\frac{1}{KT^*}\int_0^{KT^*} y_c(t)dt\right)^2+\left(\frac{1}{KT^*}\int_0^{KT^*} y_s(t)dt\right)^2 \cong P_c \qquad (3\text{-}47)$$

It remains to obtain an estimate of P_d. We may form the difference

$$\begin{aligned}
y_c(t)-y_c(t-T^*) =&\sqrt{P_d}\left[\cos\left(\theta_c+\sum_{n=-\infty}^{\infty}\theta_n p(t-nT-\varepsilon T)\right)\right.\\
&\left.-\cos\left(\theta_c+\sum_{n=-\infty}^{\infty}\theta_n p(t-T^*-nT-\varepsilon T)\right)\right]\\
&+n_c(t)-n_c(t-T^*)\\
=&\sqrt{P_d}\left\{-2\sin\left[\frac{1}{2}\left(2\theta_c+\sum_{n=-\infty}^{\infty}(\theta_n+\theta_{n+l})p(t-nT-\varepsilon T)\right)\right]\right.\\
&\left.\times\sin\left[\frac{1}{2}\sum_{n=-\infty}^{\infty}(\theta_n-\theta_{n+l})p(t-nT-\varepsilon T)\right]\right\}\\
&+n_c(t)-n_c(t-T^*)
\end{aligned}$$

where $l \triangleq T^*/T$. Although l is unknown, since T is unknown, we will see that this parameter will drop out of the final metric. Forming a similar expression for the difference of $y_s(t)$ terms, squaring, and ignoring noise terms, we obtain

$$(y_c(t) - y_c(t-T^*))^2 \cong 4P_d \Bigg\{ \sin^2\left[\frac{1}{2}\left(2\theta_c + \sum_{n=-\infty}^{\infty}(\theta_n + \theta_{n+l})p(t - nT - \varepsilon T)\right)\right] \times \sin^2\left[\frac{1}{2}\left(\sum_{n=-\infty}^{\infty}(\theta_n - \theta_{n+l})p(t - nT - \varepsilon T)\right)\right]\Bigg\}$$

$$(y_s(t) - y_s(t-T^*))^2 \cong 4P_d \Bigg\{ \cos^2\left[\frac{1}{2}\left(2\theta_c + \sum_{n=-\infty}^{\infty}(\theta_n + \theta_{n+l})p(t - nT - \varepsilon T)\right)\right] \times \sin^2\left[\frac{1}{2}\left(\sum_{n=-\infty}^{\infty}(\theta_n - \theta_{n+l})p(t - nT - \varepsilon T)\right)\right]\Bigg\}$$

Thus,

$$\frac{1}{2KT^*}\int_0^{KT^*}\left[\left(y_c(t) - y_c(t-T^*)\right)^2 + \left(y_s(t) - y_s(t-T^*)\right)^2\right] dt \cong \frac{2P_d}{KT^*}\int_0^{KT^*} \sin^2\left[\frac{1}{2}\left((\theta_n - \theta_{n+l})p(t - nT - \varepsilon T)\right)\right] dt \cong P_d \quad (3\text{-}48)$$

where we have used $\sin^2 x = (1/2)(1 - 2\cos x)$ and the fact that the integration of the cosine term approaches zero for sufficiently large K.

Thus, an ad hoc estimator of the modulation index $\beta = \cot^{-1}\sqrt{P_c/P_d}$ is given by

$$\hat{\beta} = \cot^{-1}\left[\sqrt{\frac{2\left[\left(\int_0^{KT^*} y_c(t)dt\right)^2 + \left(\int_0^{KT^*} y_s(t)dt\right)^2\right]}{KT^*\int_0^{KT^*}\left[\left(y_c(t) - y_c(t-T^*)\right)^2 + \left(y_s(t) - y_s(t-T^*)\right)^2\right] dt}}\right] \quad (3\text{-}49)$$

which is identical to Eq. (3-39) when T is replaced by T^*. Thus, the same architecture shown in Fig. 3-1 may be used when modulation type, data rate, symbol timing, and SNR are unknown, by replacing T with T^*.

3.4 Noncoherent Estimation in the Absence of Carrier Frequency Knowledge

Consider now demodulating the received signal of Eq. (3-1) with I and Q references as in Eq. (3-31) with ω_c replaced by $\omega_c - \Delta\omega$, where $\Delta\omega$ denotes the uncertainty in the knowledge of the true carrier frequency ω_c. Then, the outputs of these demodulations are given by Eq. (3-32) with θ_c replaced by $\Delta\omega t + \theta_c$. Squaring $y_c(t)$ and $y_s(t)$ and summing these squares gives

$$y_c^2(t) + y_s^2(t) =$$

$$P_c + P_d\left(\sum_{n=-\infty}^{\infty} p^2(t-nT) + \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ n\neq m}}^{\infty} a_n a_m p(t-nT)\,p(t-mT)\right) + N_1(t) \tag{3-50}$$

where as before $N_1(t)$ is composed of $S \times N$ and $N \times N$ terms. Integrating the sum in Eq. (3-50) over K symbol durations and recognizing that for sufficiently large K the time average over the data-dependent term can be replaced by the statistical average, which for random data equates to zero, we obtain the simplified result

$$\int_0^{KT}\left[y_c^2(t) + y_s^2(t)\right]dt = KT\left(P_c + P_d\right) + \overbrace{\int_0^{KT} N_1(t)\,dt}^{N_A} = KTP_t + N_A \tag{3-51}$$

Next, form the complex signal

$$\begin{aligned}\tilde{y}(t) &= y_s(t) + jy_c(t)\\ &= \left[\sqrt{P_c} + j\sqrt{P_d}\sum_{n=-\infty}^{\infty} a_n p(t-nT)\right]e^{j(\Delta\omega t + \theta_c)} + n_s(t) + jn_c(t)\end{aligned} \tag{3-52}$$

and multiply it by its complex conjugate delayed by one symbol $\tilde{y}^*(t-T)$, which gives

$$\tilde{y}(t)\,\tilde{y}^*(t-T) = \left[P_c + P_d \sum_{n=-\infty}^{\infty} a_n a_{n-1} p(t-nT)\right] e^{j\Delta\omega T}$$

$$+ j\sqrt{P_c P_d} \sum_{n=-\infty}^{\infty} (a_n - a_{n-1})\, p(t-nT) e^{j\Delta\omega T} + N_2(t) \quad \text{(3-53)}$$

where again $N_2(t)$ is composed of $S \times N$ and $N \times N$ terms. Now integrate this complex product over K symbol intervals, once again ignoring the averages over the data-dependent terms, valid for large K. Thus,

$$\int_0^{KT} \tilde{y}(t)\,\tilde{y}^*(t-T)\,dt = KTP_c e^{j\Delta\omega T} + \overbrace{\int_0^{KT} N_2(t)\,dt}^{N_B(t)} \quad \text{(3-54)}$$

Finally, from observation of Eqs. (3-51) and (3-54), it is reasonable to propose the ad hoc noncoherent estimator of modulation angle

$$\cos\beta = \sqrt{\frac{\left|\int_0^{KT} \tilde{y}(t)\,\tilde{y}^*(t-T)\,dt\right|}{\int_0^{KT} \left|\tilde{y}(t)\right|^2 dt}} \quad \text{(3-55)}$$